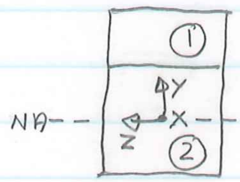
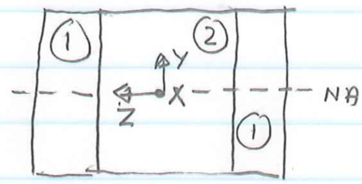
16. Beams of two materials (composite beams)
(symmetric cross-section, linearly elastic, isotropic, material 1 and material 2)

For example:



OR



showing cross-sections

We still assume plane cross-sections remain plane and the existence of a neutral axis (where $\varepsilon_x = 0$ and where Z is located). Still assume $\varepsilon_x = -y/\rho$ and only the stress component σ_x. [Note: some differences may have to be accounted for if $\nu_1 \neq \nu_2$.]

In 1, denote σ_x by σ_1 where $\sigma_1 = E_1 \varepsilon_x = -\dfrac{E_1}{\rho} y$

In 2, denote σ_x by σ_2 where $\sigma_2 = E_2 \varepsilon_x = -\dfrac{E_2}{\rho} y$

$$M = -\int_{C\text{-}S1} \sigma_1 y \, dA - \int_{C\text{-}S2} \sigma_2 y \, dA = \frac{E_1}{\rho} \int_{C\text{-}S1} y^2 dA + \frac{E_2}{\rho} \int_{C\text{-}S2} y^2 dA$$

Define $n = \dfrac{E_2}{E_1}$, then $M = \dfrac{E_1 I}{\rho}$ where $I = \displaystyle\int_{C\text{-}S1} y^2 dA + n \int_{C\text{-}S2} y^2 dA$

where I can be interpreted as the moment of inertia of a transformed cross-section where area 2 is expanded in width (in the z direction) by a factor n.

Eliminating ρ, the stresses can be expressed as

$$\sigma_1 = -\frac{My}{I} \qquad \sigma_2 = -\frac{Myn}{I}$$

To locate the neutral axis, set the resultant axial force on the cross-section equal to zero.

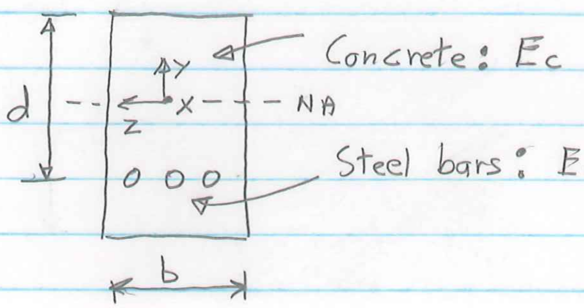
$$\int_{C-S1} \sigma_1 \, dA + \int_{C-S2} \sigma_2 \, dA = 0$$

$$-\frac{E_1}{\rho} \int_{C-S1} y \, dA - \frac{E_1 n}{\rho} \int_{C-S2} y \, dA = 0$$

$$\int_{C-S1} y \, dA + n \int_{C-S2} y \, dA = 0$$

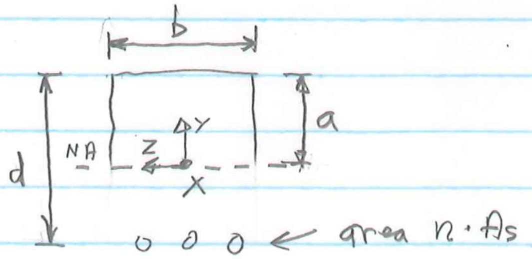
Thus, the neutral axis is the centroidal axis of the transformed cross-section.

Example: Application to reinforced concrete beams



$A_s \ll d \cdot b$

Since concrete below the neutral axis is in tension, and since concrete is weak in tension, the part of the concrete below the neutral axis will be neglected. The steel will be treated as material 2, whose area will be transformed.



where $n = \frac{E_s}{E_c}$

Find the neutral axis (find a) by setting the first moment of the area about the neutral axis equal to zero.

$$b \cdot a \cdot \frac{a}{2} - (d-a)\, n\, A_s = 0 \quad \text{(quadratic in } a\text{)}$$

Then, $I = \frac{1}{3} b a^3 + n A_s (d-a)^2$, $\sigma_c = \frac{My}{I}$, $\sigma_s = \frac{Myn}{I}$

Note: In reality, ultimate strength design is used for concrete.